Fourier Transforms

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Abstract

This is a succinct description of Fourier Transforms as used in physics and mathematics.

Fourier transforms

Defining the transforms

The formal definitions and normalizations of the Fourier transform are not standardized. We use a forward transform $\mathcal{F}$ of a function of time t and an inverse transform $\mathcal{F}^{-1}$ of a function of frequency f with a normalization and sign convention defined by Brigham ([1], pp. 48-49)

$$H(f) = \mathcal{F}(h(t)) \tag{1}$$

$$h(t) = \mathcal{F}^{-1}(H(f)) \tag{2}$$

$$H(f) = \int_{-\infty}^{+\infty} h(t) \exp(-i\, 2\pi f t) dt \tag{3}$$

$$h(t) = \int_{-\infty}^{+\infty} H(f) \exp(+i\, 2\pi f t) df \tag{4}$$

Parseval's theorem then gives

$$\int_{-\infty}^{+\infty} |h^2(t)|\, dt = \int_{-\infty}^{+\infty} |H^2(f)|\, df \tag{5}$$

Note that with these definitions for the Fourier transform pair, the frequency integration is over f rather than over $\omega = 2\,\pi f$ common in contemporary physics literature. A transformation $t \to f$ of Eq. 3 is usually referred to as a *forward* Fourier transform, and one that takes $f \to t$ of Eq. 4 is an *inverse* Fourier transform. The forward and inverse transforms are mathematically symmetric with our choice of normalization.

In contrast to this choice, the association of the + sign with the forward $t \to f$ transform is common in the recent physics literature (e.g. [2], [3], [4], [5]), in modern mathematics and programming references such as Press et al.'s *Numerical Recipes in C* [6], and in the default language of Wolfram's Mathematica [7]. However, the classic review by Wang and Uhlenbeck [8], Matlab, Scipy [9], and engineering literature use a $-$ sign for $t \to f$ and a + sign for $f \to t$. This inevitably leads to confusion, even in one application area, when the implementation of a model built with one convention is done with software that uses another.

Wolfram identifies the Fourier transform sign and normalization conventions by defining more generally

$$H(\omega) = \sqrt{\frac{|b|}{(2\pi)^{1-a}}} \int_{-\infty}^{+\infty} h(t) \exp(+ib\omega t)\, dt \tag{6}$$

suggesting these common options

Modern physics: $a = 0$ and $b = +1$

Mathematics and system engineering: $a = -1$ and $b = +1$

Classical physics: $a = +1$ and $b = -1$

Signal processing: $a = 0$ and $b = -2\pi$ (integration on $-2\pi f$)

The use of $b = \pm 1$ in this definition would lead to asymmetry between the forward and inverse transforms and is avoided by making $b = \pm 2\pi$, thus associating our choice with the "signal processing" literature. Although the + sign on b is in keeping with modern physics usage and makes comparison to contemporary physics literature simpler, the $-$ sign we have chosen matches the classical physics literature, contemporary programming standards, and applications in signal processing. Whatever the sign for the t term, spatial terms (x, y, z) should have the opposite sign so that a region of constant phase will progress in a positive spatial direction as time advances.

Examples

Consider an exponentially damped oscillator such as

$$h(t) = \exp(-a\,|t|) \cos(2\pi f_0 t) \tag{7}$$

where we have chosen to make it a symmetric function of t. Its Fourier transform is

$$\begin{aligned} H(f) &= \int_{-\infty}^{+\infty} \exp(-a\,|t|) \cos(2\pi f_0 t) \left(\cos(2\pi f t) - i\, \sin(2\pi f t)\right)\, dt && (8) \\ &= \int_{-\infty}^{+\infty} \exp(-a\,|t|) \cos(2\pi f_0 t) \cos(2\pi f t)\, dt && (9) \end{aligned}$$

which is a standard integral that evaluates to a Lorentzian

$$H(f) = 2 \cdot \frac{a}{a^2 + 4\pi^2 (f - f_0)^2} \tag{10}$$

$$= \frac{2}{a} \cdot \frac{(a/2\pi)^2}{(a/2\pi)^2 + (f - f_0)^2} \tag{11}$$

with a half width at half-maximum intensity (HWHM) $w = a/2\,\pi$, or a full width at half-maximum intensity (FWHM) $2w = a/\pi$.

The area under this Lorentzian is then

$$A = \int_{-\infty}^{+\infty} \frac{1}{\pi w} \cdot \frac{w^2}{w^2 + (f - f_0)^2} \, df \tag{12}$$

$$A = 2 \tag{13}$$

Since exponential decay and Lorentzian spectral line shapes are common in actual cases of interest, we often use these relationships to test the computation of Fourier transforms. For example, caution has to exercised when a transformation is done with a library function, since the normalization depends on the definition of the transform pairs. Indeed, the normalization may be different depending on whether a discrete or a Fast Fourier Transform (FFT) is computed. With the definitions given above, the frequency dependent $H(f)$ may be used as the argument in the inverse transform to find the original $h(t)$ without renormalizing the intermediate result.

Symmetries

Fourier transforms have useful symmetries and other properties proven in many monographs. [1, 10, 11] These are the ones given by Press et al. [6] –

$h(t)$ is real	$H(-f) = H^\star(f)$
$h(t)$ is imaginary	$H(-f) = -H^\star(f)$
$h(t)$ is even	$H(-f) = H(f)$
$h(t)$ is odd	$H(-f) = -H(f)$
$h(t)$ is real and even	$H(-f)$ is real and even
$h(t)$ is real and odd	$H(-f)$ is imaginary and odd
$h(t)$ is imaginary and even	$H(-f)$ is imaginary and even
$h(t)$ is imaginary and odd	$H(-f)$ is real and odd

$$h(at) \leftrightarrow \frac{1}{|a|} H\left(\frac{f}{a}\right) \tag{14}$$

$$\frac{1}{|b|} h\left(\frac{t}{b}\right) \leftrightarrow H(bf) \tag{15}$$

$$h(t - t_0) \leftrightarrow H(f) \exp(2\pi f t_0) \tag{16}$$

$$h(t) \exp(-2\pi f t_0) \leftrightarrow H(f - f_0) \tag{17}$$

Convolution and Correlation

When an instrument records data, the ideal signal is blurred by instrumental response. These measurements are modeled with a convolution $h*k$ of the signal h and a response function k that adds the contributions to produce the observed signal. It is computed with the integral

$$h * k(\tau) = \int_{-\infty}^{+\infty} h(t)\, k(\tau - t)\, dt \tag{18}$$

Although the source and response functions in the convolution are conceptually different, the result is symmetric on exchange of their roles, that is,

$$h * k = k * h \tag{19}$$

The Convolution Theorem states that the Fourier transform (represented by $\mathcal{F}$) of a convolution is the product of Fourier transforms of the two functions

$$\mathcal{F}(h * k) = \mathcal{F}(h) \cdot \mathcal{F}(k) \tag{20}$$

and that

$$\mathcal{F}(h \cdot k) = \mathcal{F}(h) * \mathcal{F}(k) \tag{21}$$

This raises the possibility of inverting a convolution, or *deconvolving* a signal, by dividing its Fourier transform by the Fourier transform of the instrumental response. Formally, we could find

$$h(\tau) = \mathcal{F}^{-1}\left(\mathcal{F}(h * k) / \mathcal{F}(k)\right) \tag{22}$$

from the signal $h * k$ if we knew k and could compute its Fourier transform $\mathcal{F}(k)$. The method works in principle as long as $\mathcal{F}(k) \neq 0$. However, noise in the unfiltered signal may introduce meaningless effects in its deconvolution.

The *convolution* function of one variable is closely related to a *correlation* of the signal here and now to one later and "over there". We symbolically let $\Gamma(h, k; p_0, t_0, p_1, t_1)$ represent these dependencies for the general case of space-time cross correlation. More simply, with only time dependence, a correlation function would compare data at the same point differing by a delay τ. For two complex functions the cross correlation is given by

$$\Gamma(h, k; \tau) = (h^\star * k)(\tau) \tag{23}$$

$$= \int_{-\infty}^{+\infty} h^\star(t)\, k(t + \tau)\, dt \tag{24}$$

It is sometimes given equivalently as

$$\begin{aligned}\Gamma(h,k;\tau) &= \int_{-\infty}^{+\infty} (h(t-\tau))^{\star}\, k(t)\, dt &(25)\\ &= ((h^{\star} * k)(-\tau))^{\star} &(26)\end{aligned}$$

The correlation in time given by Eq. 24 and the convolution of two temporal signals given by Eq. 18 are consistent with the astronomical ([3], p.39) and optical ([2], p. 564) literature. These concepts are also widely used in digital signal processing [11, 12] and numerical analysis [6].

The correlation of a function with itself is the autocorrelation

$$\Gamma(h,h) = \int_{-\infty}^{+\infty} h^{\star}(t)\, h(t+\tau)\, dt \tag{27}$$

In the theory of stationary random processes, the Wiener-Khinchine theorem provides that Fourier transform of the autocorrelation of a function is its power spectrum ([2], p. 568)

$$\mathcal{F}(\Gamma(h,h)) = (\mathcal{F}(h))^{\star} \cdot \mathcal{F}(h) \tag{28}$$

Useful definite integrals

The following definite integrals are useful in testing numerical evaluations of Fourier transforms and correlation functions.

A Lorentzian arises from the Fourier transform of a exponential

$$\begin{aligned}\int_0^{\infty} e^{-ax}\, \cos(bx)\, dx &= \left(\tfrac{a}{a^2+b^2}\right) &(29)\\ \int_0^{\infty} e^{-ax}\, \sin(bx)\, dx &= \left(\tfrac{b}{a^2+b^2}\right) &(30)\end{aligned}$$

The area under this Lorentzian is

$$\int_{-\infty}^{+\infty} \left(\frac{a}{a^2+x^2}\right) dx = \pi \tag{31}$$

The area-normalized Lorentzian is

$$L(x) = \left(\frac{1}{a\pi}\right)\left(\frac{a^2}{a^2+x^2}\right) \tag{32}$$

A Gaussian arises from the Fourier transform of a Gaussian

$$\int_{-\infty}^{+\infty} \exp\left(-ax^2\right)\, \cos(bx)\, dx = \sqrt{\pi/a}\, \exp\left(-b^2/4a\right) \tag{33}$$

The area under a Gaussian is

$$\int_{-\infty}^{+\infty} e^{-ax^2}\, dx = \sqrt{\pi/a} \tag{34}$$

A normal distribution is a Gaussian of unit area with mean μ, standard deviation σ, and variance σ^2

$$N(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \tag{35}$$

The integral of x^2 under a Gaussian is

$$\int_{-\infty}^{+\infty} x^2 \exp(-ax^2)\, dx = \frac{\sqrt{\pi}}{2}\, a^{-3/2} \tag{36}$$

Consequently, the volume integral on r from 0 to ∞ is

$$\int_{0}^{+\infty} 4\pi\, r^2\, \exp(-ar^2)\, dr = \pi\, \sqrt{\pi}\, a^{-3/2} \tag{37}$$

The volume-normalized Gaussian density is therefore

$$G(r) = \left(\frac{a}{\pi}\right)^{3/2} \exp(-ar^2) \tag{38}$$

such that

$$\int_{-\infty}^{+\infty} G(r)\, 4\pi\, r^2\, dr = 1 \tag{39}$$

The Airy function, A_i, is defined by

$$\int_{0}^{\infty} \cos(at^3 \pm xt)\, dt = (3a)^{-1/3}\, \pi\, A_i\left[\pm(3a)^{-1/3}\, x\right] \tag{40}$$

The exponential integrals E_i, S_i and C_i are

$$E_i(x) = \int_{-\infty}^{x} \left(\frac{e^{+it}}{t}\right) dt = -\int_{-x}^{\infty} \left(\frac{e^{-it}}{t}\right) dt \tag{41}$$

$$S_i(x) = -\int_{-x}^{\infty} \left(\frac{\sin(t)}{t}\right) dt \tag{42}$$

$$C_i(x) = -\int_{-x}^{\infty} \left(\frac{\cos(t)}{t}\right) dt \tag{43}$$

$$E_i(x) = C_i(x) + iS_i(x) \tag{44}$$

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